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PARTITIONING POINT SETS IN 4 DIMENSIONS

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PARTITIONING POINT SETS IN 4 DIMENSIONS

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ABSTRACT

We introduce a new type of partition called a parallel planes partition. We prove there exists a parallel planes partition of any set of *n* points in 4 dimensions. This partition yields a data structure for the half-space retrieval problem in 4 dimensions; it has linear size and achieves a sublinear query time.

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1. Introduction

The half-space retrieval problem is the following. Given a set of n points in d-dimensional Euclidean space, preprocess them so as to be able to answer queries of the following form: how many points lie in a given half space. (A variant of the problem, the *listing problem*, is to ask for a list of the points in the half space.) It is assumed that many such queries will be made. Thus it is reasonable to preprocess the set of points and to amortize the cost of this preprocessing over the (many) queries. What we are concerned with is the query time and the space used by the data structure for holding the preprocessed information. We note that a naive searching algorithm takes linear time. Thus we aim for a sublinear search time and a data structure using linear space.

Recently, an elegant approach to this problem was discovered by Willard [W]. There are two basic lemmas underlying his construction. First, a set of n points can be partitioned by two straight lines into 4 sets, so that each (open) set holds at most $n/4$ points. This leads to a recursive storage of the points in a 4-way balanced tree. Second, any line only intersects 3 of the 4 sets defined by the two partitioning lines. This implies that in carrying out a half-space query at most three of the four subtrees of the root need be explored further. Recursive application of this observation yields a sublinear search time. (it is convenient to speak of a line, or plane, *missing* a set S when the line, or plane, does not intersect S.)

By tuning his data structure (dividing the set into 6 equal parts with 3 lines, rather than 4 parts with 2 lines) Willard obtained a linear-sized data structure which allowed a query time of $O(n^{0.77})$. (Henceforth we will not refer to the size of the data structure, for it is always linear in this paper.) Subsequent work by Edelsbrunner and Welzl [EW] improved the search time to $O(n^{0.695})$.

Using the same approach Yao [Y1,Y2] obtained a similar result in 3 dimensions. She proved that any set of n points can be divided by 3 planes into 8 equal sized sets. This yields an algorithm for the half-space retrieval problem with a sublinear query time. Dobkin and Edelsbrunner [DE1,E] then further improved the algorithm that is derived from this partition, obtaining a search time of $O(n^{0.89})$.

In d-dimensions, we seek to partition the set of n points into 2^d subsets, using d hyperplanes, so that each subset contains at least αn points, for some constant $0 \leq \alpha \leq 2^d$. Such a partition would yield an algorithm for half space retrieval, with sublinear search time, similar to Willard's. Avis showed that for $d \geq 5$ there exist sets of n points for which there is no such partition. While not proving that there is no algorithm with a sublinear search time, this result does have a somewhat negative flavor.

We show a partitioning result for sets of n points in 4 dimensions. We show that there is a partition (which we call a parallel planes partition) such that each of the 16 sets defined by the partition contains at least $n/32$ points (loosely speaking; we are more precise in the next section). This partition immediately yields an algorithm for the half-space retrieval problem in 4 dimensions with a query time of $O(n^{0.93})$. Further applications of the partition result include the circle retrieval problem [Y] and other query problems [DE2]. Our work extends these results to one higher dimension; for example, our partition implies that there is an algorithm for the sphere retrieval problem with a sublinear query time.

An interesting aspect of our work is that the partition is not by 4 planes (in fact it uses parts of 8 planes). So the result proved by Avis is not applicable to our construction. This opens up the prospect of finding similar partitions for point sets in 5 or more dimensions. Also, it suggests one might look for other partitions enjoying the following properties:

(a) Every set in the partition should contain some constant fraction of the points.

(b) Any plane misses at least one of the sets defined by the partition.

We call such a partition a *separating partition*. A separating partition immediately yields an algorithm for half space retrieval, of the form given by Willard, having a sublinear search time. A caveat should be made here. In Willard's data structure it is necessary to be able to perform half space queries on the separating lines, if they contain any of the points. Similarly, to obtain an efficient algorithm from a class of separating partitions, we need to be able to perform half space queries on the separating surfaces, and these queries must be performed as fast as the general queries. (Our partition uses 3 dimensional planes for the separating surfaces, and for these surfaces the requirement is met.)

There has been considerable related work in this area. The data structures we have been describing are static; Fredman has proved lower bounds on the complexity of dynamic data structures for this problem [F]. When considering the listing problem we often use a different measure for the query time, $Q(n)$. $Q(n)$ is chosen so that the time taken to report the s points in a half space is $O(s+Q(n))$. The goal has been to achieve a fast query time $Q(n)$, while dropping the requirement for the data structure to have linear size [CY,CGL]. We will not consider these problems further, however.

In the next section we describe the form of our partition, prove that it is separating, and deduce the search time for the resulting algorithm. In section 3 we prove the partition exists.

2. The Partition is Separating

We make the notion of a partition precise.

Definition: [Y1] Let S be a set of n points in d-dimensional space. A *partition* Π for S is a pair $(\mathbf{R},\mathbf{P})$, where $\mathbf{R}$ is a set of disjoint open regions and $\mathbf{P}$ is a set of $(d-1)$-dimensional surfaces, such that every point of S either lies in a region $R \in \mathbf{R}$, or on a surface $P \in \mathbf{P}$. A region $R \in \mathbf{R}$ is said to be a δn*−region* with respect to Π if at least δn points of S are contained in $R \bigcup_{P \in \mathbf{P}} P$ (i.e. the remaining open regions of $\mathbf{R}$ contain at most $(1-\delta)n$ points). We say $\Pi = (\mathbf{R},\mathbf{P})$ is a δn*−partition* if every $R \in \mathbf{R}$ is a δn-region.

We describe our partition next. As an aid to our intuition, we start by giving a partition Π of 3 dimensional space. We have 4 partitioning planes: P, Q, R_1, R_2 (see figure 1). R_1 and R_2 are parallel; further, we only use parts of the planes R_1 and R_2, as illustrated in figure 1. It is clear these surfaces define 8 disjoint, open regions.

Lemma 1: Any plane, M, intersects at most seven of the regions defined by Π.

Proof: Let L be the intersection of M with P. Without loss of generality, essentially only two situations arise, as illustrated in figure 2. In case (a), since L does not intersect either A or B, M cannot intersect one of U_1 and U_3. In case (b), since L does not intersect C or D, M does not intersect one of U_3 and U_5. □

The partition Π in 4 dimensions uses 8 planes: P, Q, R_1, R_2, S_1, S_2, S_3, S_4. R_1 and R_2 are parallel, as are S_1, S_2, S_3, and S_4. P, Q, R_1, and R_2 have the same form as in the 3 dimensional case; let us label the 8 resulting regions, as before, as shown in figure 1 (we also label the four regions defined by P and Q as shown in figure 1). We keep just those parts of S_i that intersect the regions U_i and U_{4+i}, $1 \leq i \leq 4$; henceforth, when we refer to S_i (or R_j) we intend the surface consisting of the parts we have kept. Let W_{2j} and W_{2j-1} be the two regions obtained from partitioning U_j by S_i ($j = i, 4+i$), $1 \leq i \leq 4$.

Lemma 2: Any 3 dimensional hyperplane M intersects at most 15 of the 16 regions defined by Π.

Proof: It is helpful to make a definition. We call $\overline{W} \cap P$ the *floor* of a region W with respect to a lower dimensional surface P (i.e. that part of P on which W stands). We show that there are two regions D_1 and D_2, in opposite quadrants (defined by P and Q), such that M intersects neither of them, and such that one of them contains a region W_l.

Consider $P \cap Q$, illustrated in figure 3. M intersects at most 7 of the regions B_l. Let the two sets of regions not intersected by M be denoted C_1 and C_2. (In the example $C_1 = B_6 \cup B_1 \cup B_2 \cup B_3 \cup B_4$ and $C_2 = B_{13} \cup B_{14} \cup B_{15}$; it may be that one of C_1 or C_2 is empty.) Consider the four quadrants defined by P and Q. In one of these quadrants M does not intersect the region D_1 standing on C_1. In the opposite quadrant M does not intersect the region D_2 standing on C_2.

By construction, only one of R_1 and R_2 intersects these two quadrants. Without loss of generality suppose that it is R_1 that intersects these two quadrants. Then S_1 and S_2 are the other planes intersecting these quadrants (S_1 partitions U_1 and U_5, and S_2 partitions U_2 and U_6). In figure 4, we show the correspondence between the regions W_i and B_j (that is, on what region in $P \cap Q$ each W_i stands). Without loss of generality we have one of the following two cases. (This is really the same situation as in lemma 1.)

(a) $B_1 \cup B_2 \subseteq C_1$; then W_1 and W_{11} both stand on a subset of C_1 and so M does not intersect one of them.

(b) $B_1 \subseteq C_1$ and $B_8 \cup B_9 \cup B_{10} \cup B_{13} \cup B_{14} \cup B_{15} \subseteq C_2$; then M does not intersect one of W_1 and W_4. □

In the next section we show there exists such an $n/32$-partition; we call it a *parallel planes partition*. We deduce

Lemma 3: There exists a linear size data structure for half space retrieval in 4 dimensions which allows a query time of $O(n^{0.93})$.

Proof: The search structure is built as described in Willard's paper, and as outlined in the introduction. A bound $T(n)$, for the search time is given by the following recurrence.

$$T(n) = \max\ [c + \sum_{i=1}^{15} T(n_i) + \sum_{i=1}^{8} S(m_i)], \quad n \geq 2,$$

$$T(1) = c$$

where $\sum_{i=1}^{15} n_i \leq 31n/32$, $\sum_{i=1}^{15} n_i + \sum_{i=1}^{8} m_i \leq n$, $n/32 \leq n_i \leq 3n/32$, $S(m_i)$ is the time to carry out a half space query on a data structure for a 3 dimensional space containing m_i points, and c is a constant.

The right hand side is largest when $n_i = 31n/15 \cdot 32$ and $m_i = n/8 \cdot 32$, giving the solution

$T(n) = 15T(31n/15\cdot 32) + O(n^{0.89})$, or

$T(n) = O(n^{\alpha})$, where $\alpha = \dfrac{\log 15}{\log 32\cdot 15/31}$

that is, $T(n) = O(n^{0.93})$.

3. Existence of the Partition

We show there is an $n/32$ parallel planes partition of any set S of n points in 4 dimensions. This section provides a proof of existence and not an algorithm. There are two major concerns in our construction. The first is to select each plane in such a way that it is unique. The second concern arises because our construction involves certain parameters which may vary. The concern is to ensure that the planes vary continuously with these parameters.

It is simpler to carry out our construction on a connected body for then we can use theorems from analysis. So we extend the definition of a δn-partition in the natural way to a set consisting of one, or several continuous bodies of total mass n. (The formal details are left to the reader.) As described in [DE1] we replace each point by a sphere of radius ϵ centered at that point; the sphere is of uniform density and has mass 1. ϵ is chosen sufficiently small so that if a plane intersects k of the spheres then there is a plane intersecting the corresponding k points. Call the new set the *sphere set*.

Lemma 4: [DE1] If there is a δn-partition Π' of the sphere set then there is a δn-partition Π of the point set S.

We transform the set further. Let B be a (large) rectangular box containing the spheres. The space in B, not occupied by spheres, is filled with matter of uniform density and total mass m ($<<1$). m is chosen so that $\lceil \delta n - m \rceil = \lceil \delta n \rceil$. (in our construction $\delta = 1/32$, so that $m = 1/100$ suffices.) Call this the *continuous set*.

Lemma 5: If there is a $\delta(n+m)$-partition Π' of the continuous set then there is a δn-partition of the point set S.

Proof: Clearly a $\delta(n+m)$-partition of the continuous set implies a $[\delta n-(1-\delta)m]$-partition of the sphere set, and hence of the point set S. On a point set this is a $(\lceil \delta n-(1-\delta)m \rceil)$-partition, which by assumption is a δn-partition. □

Thus to obtain our result it suffices to show that a body S, of mass n, has an $n/32$-partition. We use the following two lemmas in our construction.

Lemma 6: [W] Given a connected body S of mass n, in two dimensional Euclidean space, and given a line L forming an $n/2$-partition of S, there is a unique line K such that L and K form an $n/4$ partition. Furthermore, K varies continuously with L.

Willard did not prove exactly this theorem, but the proof would be identical to Willard's. The fact that S is connected guarantees uniqueness.

Lemma 7: (Borsuk-Ulum theorem [L]). Let f be a continuous function, $f\colon \mathbf{S}^d \to \mathbf{R}^d$. Then there is an $\underline{x}$ such that $f(\underline{x}) = f(-\underline{x})$.

Definition: f is an *even* function if for all $\underline{x}$, $f(\underline{x}) = -f(-\underline{x})$.

Corollary: If f is an even, continuous function, $f\colon \mathbf{S}^d \to \mathbf{R}^d$, then there is an $\underline{x}$ such that $f(\underline{x}) = f(-\underline{x}) = \underline{0}$.

We build the planes described in the previous section. At times we will project onto 2 or 3 dimensional spaces; where no ambiguity will result we will name by P the projection of P, etc. We construct P, Q, R_i, S_j, in turn.

We choose P to be the bisector of S normal to $\underline{w}$, a direction vector. We note P is unique, varies continuously with $\underline{w}$, and is reversed when $\underline{w}$ is reversed. We define the other planes, in part, in terms of $\underline{w}$ and a set of coordinate axes on P. Since we are going to vary $\underline{w}$, and we wish the coordinate axes to vary continuously, we need to show how to define the axes in terms of some original set of axes.

Let $\underline{w}$, $\underline{x}$, $\underline{y}$, $\underline{z}$ be the 4 mutually perpendicular axes. And let $\underline{w_0}$, $\underline{x_0}$, $\underline{y_0}$, and $\underline{z_0}$ be some initial set of mutually perpendicular axes. Suppose $\underline{w} = (a,b,c,d)$ (i.e. $\underline{w} = a\underline{w_0}+b\underline{x_0}+c\underline{y_0}+d\underline{z_0}$). Then we define $\underline{x}$, $\underline{y}$, $\underline{z}$ to be, respectively, $(-b,a,-d,c)$, $(-c,d,a,-b)$, $(-d,-c,b,a)$. It is easy to check the 4 axes are perpendicular and that they vary continuously with $\underline{w}$. (We were guided by the definition of quaternions and we are indebted to Colm O'Dunlaing for suggesting this approach.)

It is convenient to use spherical coordinates to refer to the orientation of $\underline{w}$ with respect to the axis system $\underline{w_0}$, $\underline{x_0}$, $\underline{y_0}$, $\underline{z_0}$; let (θ,ϕ,ψ) be this orientation. It is clear P is unique and varies continuously with (θ,ϕ,ψ), since S is a connected body.

We choose Q to intersect P parallel to the $\underline{x}$ and $\underline{y}$ axes. In addition, we would like P and Q to form an $n/4$-partition. By projecting parallel to $\underline{x}$ and $\underline{y}$, this becomes the 2 dimensional problem described in lemma 6. We deduce such a Q exists and is unique. Further Q varies continuously with (θ,ϕ,ψ), and is reversed when this orientation is reversed.

We choose R_1 and R_2 so that P, Q, R_1 and R_2 form an $n/16$-partition. Before describing our choice of R_1 and R_2 it is convenient to give another definition and to prove a lemma. We generalize the notion of a k-hull/center introduced in [CSY,Y1].

Definition: The δm-hull of a body of mass m consists of the set of points p, such that any plane through p has mass $\geq \delta m$ to either side.

Lemma 8: A closed, compact body B of mass m in $\mathbf{R}^{d-1}$ has a non-empty m/d-hull.

Proof: We first show that the body has a non-empty $(m/d-\epsilon)$-hull, H_ϵ, for all $\epsilon>0$. We also show H_ϵ is compact. Since $H_\epsilon \supset H_\delta$, for $\epsilon>\delta$, we deduce $\bigcap_{\epsilon>0} H_\epsilon \neq \varnothing$. But $\bigcap_{\epsilon>0} H_\epsilon$ is the m/d-hull.

We show H_ϵ is non-empty. Consider the set Π consisting of the half spaces S, where S is defined by a $(d-1)$-dimensional plane, such that the mass of that part of the body in S is exactly

$m/d+\epsilon$. Then the intersection of d of these spaces contains mass $\geq 5\epsilon$, and thus is non-empty. By Helly's theorem [YB], we deduce the intersection of all the half spaces is non-empty. But this intersection is contained in H_ϵ; hence H_ϵ is non-empty.

We now show H_ϵ is compact. We observe H_ϵ is closed (from the definition of ϵm-hulls). In addition, H_ϵ is contained in B, a compact set. So H_ϵ is compact. □

We are able to reduce the choice of R_1 and R_2 to a 3 dimensional problem. We do this by choosing R_1 and R_2 to intersect P parallel to the $\underline{x}$ axis. We then project parallel to the $\underline{x}$ axis. Recall R_1 and R_2 are parallel. Let K_i be the $n/4$-hull of T_i. Let c_i be the center of gravity of K_i, considered as a body of uniform density. Let $\underline{l}_1$ and $\underline{l}_2$ be the vectors parallel to the lines from c_1 to c_3 and from c_2 to c_4, respectively. We choose both R_1 and R_3 to be parallel to both $\underline{l}_1$ and $\underline{l}_2$; also we choose R_1 to pass through c_1 and c_3, while we choose R_2 to pass through c_2 and c_4. We note $\underline{l}_1$ and $\underline{l}_2$ are not parallel (for if they were, by projecting parallel to the $\underline{y}$-axis, which is not parallel to either $\underline{l}_1$ nor $\underline{l}_2$, we obtain the situation shown in figure 5, with $\underline{l}_1$ and $\underline{l}_2$ ostensibly parallel, a contradiction). Thus R_1 and R_2 are well defined planes. Clearly they are unique. Since the sets K_i (and thus the points c_i) vary continuously with (θ,ϕ,ψ), we deduce that R_1 and R_2 vary continuously with these angles also. Also R_1 and R_2 are reversed when (θ,ϕ,ψ) is reversed. Because R_1 and R_2 pass through c_1 and c_3, c_2 and c_4, respectively, we know that each set U_i, $1 \leq i \leq 8$, has mass $\geq m/16$, and $\leq 3m/16$.

It remains to define the planes S_i. Recall that these planes are required to be parallel. So let $\underline{s}$ be the normal to the planes S_i. Let ρ be the angle between $\underline{s}$ and $\underline{r}$, the normal to the planes R_1 and R_2. By choosing the angles (ρ,θ,ϕ,ψ) appropriately, we aim to obtain planes S_i, such that S_i bisects both the sets U_i and U_{4+i}, $1 \leq i \leq 4$. It is convenient, in defining the planes S_i to normalize so that the sets U_j each have mass 1. We then define S_i to be the unique plane bisecting $U_i \cup U_{4+i}$, $1 \leq i \leq 4$.

We have yet to explain how $\underline{s}$ is chosen. We want to choose $\underline{s}$ uniquely, given $\underline{r}$ and ρ; we achieve this by choosing $\underline{s}$ to lie in a plane determined by $\underline{r}$. Let $\underline{w}'=\underline{r}$, and suppose $\underline{x}'$, $\underline{y}'$, $\underline{z}'$ are defined in terms of $\underline{w}_0$, $\underline{x}_0$, $\underline{y}_0$, $\underline{z}_0$, in the same way as $\underline{x}$, $\underline{y}$, and $\underline{z}$ were, given $\underline{w}$. Then we restrict $\underline{s}$ to lie in the $\underline{w}'-\underline{z}'$-plane. So $\underline{s}$ is uniquely defined by $\underline{r}$ and ρ. Also, $\underline{s}$ varies continuously with (ρ,θ,ϕ,ψ), and hence so do the planes S_i, $1 \le i \le 4$. When (θ,ϕ,ψ) is reversed, since both $\underline{r}$ and the coordinate axes are reversed, $\underline{s}$ is also reversed. When ρ is also reversed, $\underline{s}$ is reversed again, and thus returns to its original orientation. Thus, when (ρ,θ,ϕ,ψ) is reversed, the planes P, Q, R_1, and R_2 are all reversed, while the planes S_i are unchanged.

We show how to choose the angles (ρ,θ,ϕ,ψ) so that each of the S_i are bisectors of the two sets U_i and U_{4+i}, $1 \le i \le 4$. We define a function $f(\rho,\theta,\phi,\psi)$, $f\colon \mathbf{S}^4 \to \mathbf{R}^4$, with f continuous and even. Thus we can apply the corollary to the Borsuk-Ulum theorem. f is chosen in such a way that when $f(\underline{x}) = \underline{0}$, the planes S_i bisect the sets U_j as required.

f is defined to be the vector (f_1,f_2,f_3,f_4); the f_i are defined below. Let S_i divide U_j ($= U_i$ or U_{4+i}) into W_{2j-1} and W_{2j}, with W_{2j} on the positive side of the dividing plane. Let t denote the (normalized) mass of the set T; by a normalized mass we intend the mass of the set T, where the mass of each set U_j is normalized to be equal to 1. We define $f_1(\underline{x}) = w_1(\underline{x}) - w_9(\underline{x})$, $f_2(\underline{x}) = w_3(\underline{x}) - w_{11}(\underline{x})$, $f_3(\underline{x}) = w_5(\underline{x}) - w_{13}(\underline{x})$, and $f_4(\underline{x}) = w_7(\underline{x}) - w_{15}(\underline{x})$.

Lemma 9: f is an even, continuous function.

Proof: It is clear f is continuous. We show $f_1(\underline{x}) = -f_1(-\underline{x})$. We note $w_1 + w_9 = w_2 + w_{10}$ (since S_1 bisects $U_1 \cup U_5$), and $w_1 + w_2 = w_9 + w_{10}$ (since $t_1 = t_3$). So $w_1 = w_{10}$ and $w_2 = w_9$. As observed above, when the orientation $\underline{x} = (\rho,\theta,\phi,\psi)$ is reversed the planes P, Q, R_1, and R_2 are all reversed, while the planes S_i have the same orientation. Hence $U_i(-\underline{x}) = U_{i+4}(\underline{x})$ and $U_{i+4}(-\underline{x}) = U_i(\underline{x})$, $1 \le i \le 4$, while $W_i(-\underline{x}) = W_{i+8}(\underline{x})$ and $W_{i+8}(-\underline{x}) = W_i(\underline{x})$, $1 \le i \le 8$. It follows that $f_1(-\underline{x}) = w_1(-\underline{x}) - w_9(-\underline{x}) = w_9(\underline{x}) - w_1(\underline{x}) = -f_1(\underline{x})$; similarly for f_2, f_3, and f_4. We deduce f is even. □

When $\rho = 0$ or π, $\underline{r}$ and $\underline{s}$ are parallel, so f remains constant despite any variation of (θ,ϕ,ψ). Hence $f: \mathbf{S}^4 \to \mathbf{R}^4$. Applying the corollary to the Borsuk-Ulum theorem we deduce there is a value $\underline{x}$ such that $f(\underline{x}) = \underline{0}$. But then $f_1(\underline{x}) = 0$ so $w_1 = w_9$, which implies $w_1 = w_2$. Similarly, all the other sets U_j are bisected. Since the mass of U_j lies between $m/16$ and $3m/16$, it follows that the mass of each W_i lies between $n/32$ and $3n/32$. We deduce

Theorem: The planes P, Q, R_i, S_j, $1 \leq i \leq 2$, $1 \leq j \leq 4$, form an $n/32$-partition.

We briefly discuss an algorithm for finding a parallel planes partition. It is clear that any parallel planes partition can be transformed while maintaining the following two properties:

(a) The points in the closure of each region defined by the partition are unchanged.

(b) Each of the following planes passes through at least 4 points: P, Q, at least one of the R_i, and at least one of the S_j.

This implies that there are only $O(n^{16})$ possibilities to check, thereby giving a polynomial time algorithm. It seems likely that there should be a more efficient algorithm. One possibility is to try to apply Megiddo's technique [M], as illustrated in [CSY].

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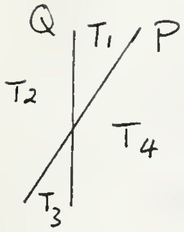


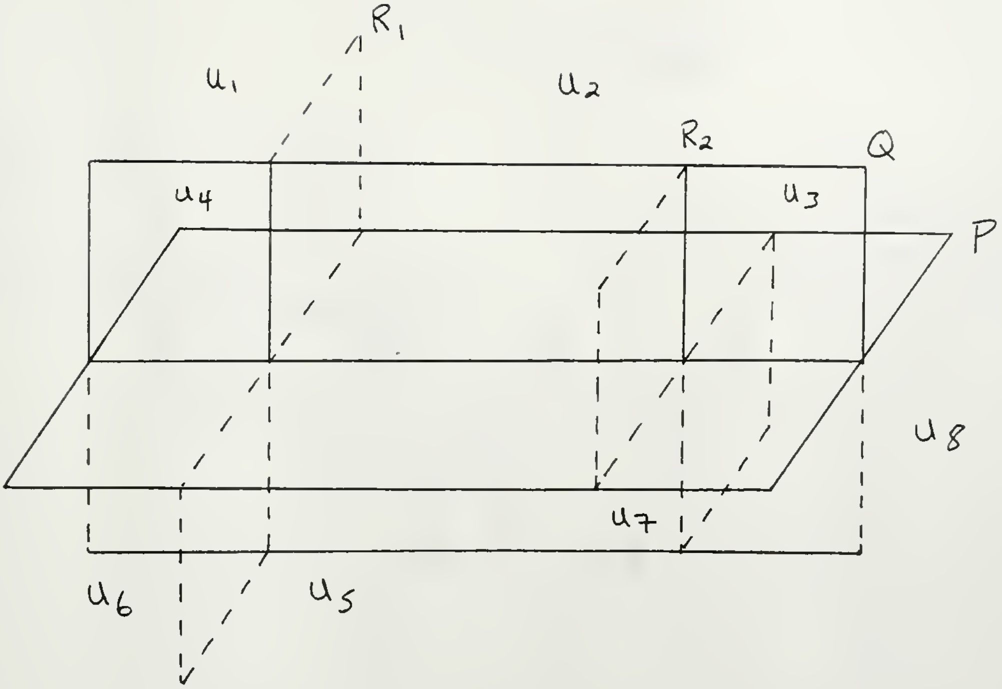
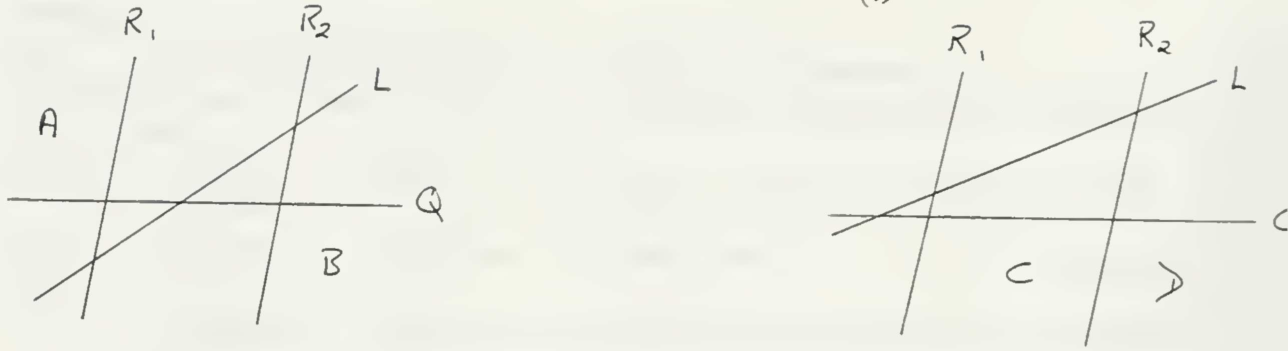


Figure 1


Either (a)

Or (b)

Figure 2

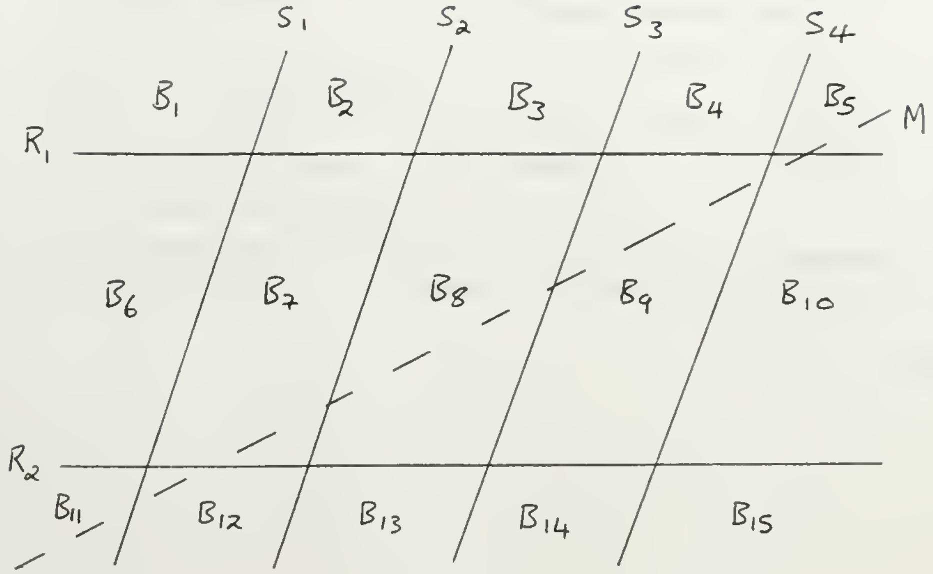
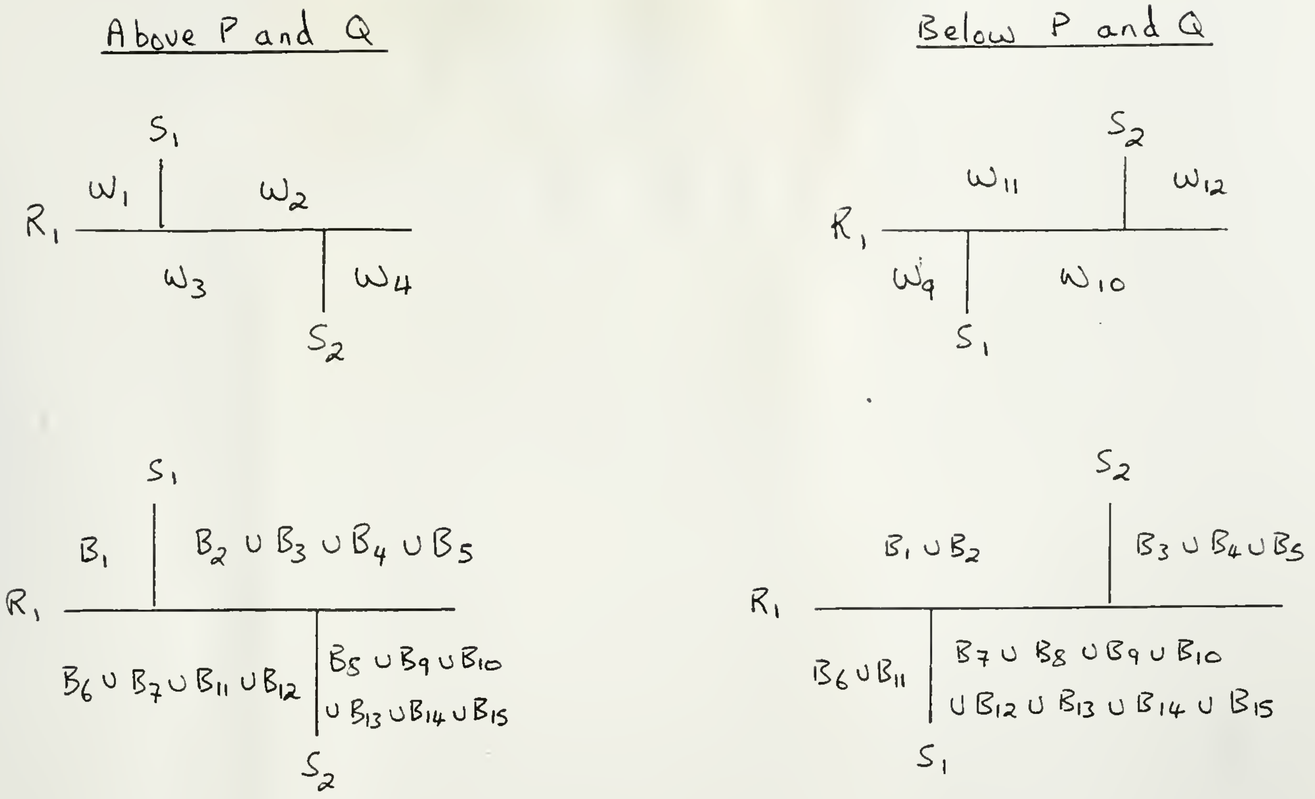


Figure 3


Above P and Q

Below P and Q

S_1
ω_1
ω_2
R_1
ω_3
ω_4
S_2
S_2
ω_{11}
ω_{12}
R_1
ω_9
ω_{10}
S_1
S_1
B_1
$B_2 \cup B_3 \cup B_4 \cup B_5$
R_1
$B_6 \cup B_7 \cup B_{11} \cup B_{12}$
$B_8 \cup B_9 \cup B_{10} \cup B_{13} \cup B_{14} \cup B_{15}$
S_2
S_2
$B_1 \cup B_2$
$B_3 \cup B_4 \cup B_5$
R_1
$B_6 \cup B_{11}$
$B_7 \cup B_8 \cup B_9 \cup B_{10} \cup B_{12} \cup B_{13} \cup B_{14} \cup B_{15}$
S_1

Figure 4

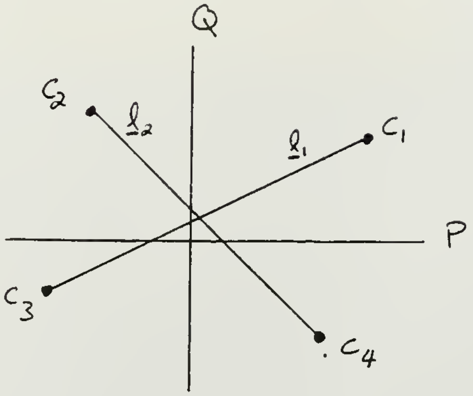


Figure 5

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